Journal of Computational and Applied Mathematics 236 (2012) 3751–3762

Contents lists available at SciVerse ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

Linear bilevel programming with interval coefficients[☆]

Herminia I. Calvete[a,*], Carmen Galé[b]

[a] Dpto. de Métodos Estadísticos, Universidad de Zaragoza, Pedro Cerbuna 12, 50009 Zaragoza, Spain

[b] Dpto. de Métodos Estadísticos, Universidad de Zaragoza, María de Luna 3, 50018 Zaragoza, Spain

ARTICLE INFO

Article history:
Received 30 September 2010
Received in revised form 13 October 2011

MSC:
90C26
90C30

Keywords:
Linear bilevel programming
Uncertainty
Interval coefficients
Sensitivity analysis

ABSTRACT

In this paper, we address linear bilevel programs when the coefficients of both objective functions are interval numbers. The focus is on the optimal value range problem which consists of computing the best and worst optimal objective function values and determining the settings of the interval coefficients which provide these values. We prove by examples that, in general, there is no precise way of systematizing the specific values of the interval coefficients that can be used to compute the best and worst possible optimal solutions. Taking into account the properties of linear bilevel problems, we prove that these two optimal solutions occur at extreme points of the polyhedron defined by the common constraints. Moreover, we develop two algorithms based on ranking extreme points that allow us to compute them as well as determining settings of the interval coefficients which provide the optimal value range.

1. Introduction

Bilevel programming involves two optimization problems where the constraint region of one of the problems is implicitly determined by the other. Bilevel problems have been proposed for dealing with hierarchical processes involving two levels of decision making and have been increasingly addressed in the literature. They can be formulated as follows:

$$\begin{array}{lll} \min\limits_{x,y} & f_1(x,y), & \text{where } y \text{ solves} \\ \min\limits_{y} & f_2(x,y) & \\ \text{s.t.} & (x,y) \in S & \end{array} \tag{1}$$

where $x \in \mathbb{R}^{n_1}$ are the upper level variables, which are controlled by the upper level decision maker; $y \in \mathbb{R}^{n_2}$ are the lower level variables, which are controlled by the lower level decision maker; $f_1, f_2 : \mathbb{R}^{n_1+n_2} \longrightarrow \mathbb{R}$ are the upper level and lower level objective functions, respectively; $S \subseteq \mathbb{R}^{n_1+n_2}$ is the common constraint region. Due to their structure, bilevel programs are nonconvex and quite difficult to deal with and solve. In fact, even the simplest model in bilevel programming, the linear bilevel program, in which all functions involved are linear, is strongly NP-hard [1]. Colson et al. [2], Dempe [3] and Vicente and Calamai [4] provide surveys on the subject. Bard [5], Dempe [6] and Shimizu et al. [7] are good textbooks on this topic.

In the above mathematical formulation of the problem, the coefficients are assumed to be known exactly. However, in practice, it is very common for the coefficient values to be only approximately known. When there is one single level of decision making, several approaches have been proposed in the literature to describe and treat imprecise and uncertain elements present in decision problems. Fuzzy programming and stochastic programming are frequently used to tackle the

☆ This research work has been supported by the Spanish Ministry of Education and Science under grant MTM2007-66893.

* Corresponding author. Tel.: +34 976762210; fax: +34 976761115.
E-mail addresses: herminia@unizar.es (H.I. Calvete), cgale@unizar.es (C. Galé).

0377-0427/$ – see front matter © 2011 Elsevier B.V. All rights reserved.
doi:10.1016/j.cam.2011.10.012

problem of inexactness in coefficients. The former assumes that membership functions of fuzzy parameters are known. The latter requires probability distributions of coefficients to be known. However, in some real-world cases it might be difficult for decision makers to specify any of these assumptions.

To overcome these difficulties, mathematical programs whose parameters are only assumed to be in intervals have been considered in the literature, almost exclusively in linear programming. Some references consider only interval numbers in the objective function coefficients of a linear problem. Among them, Steuer [8] proposes algorithms for determining all extreme points and unbounded edge directions that solve the problem for at least one setting of the coefficient values in their ranges. Taking a different approach, Ishibuchi and Tanaka [9] assume that decision makers have preferences when selecting intervals. They define several partial order relations which represent the decision maker's preferences between intervals and propose their use for selecting optimal solutions in minimum linear programs with interval coefficients in the objective function. Finally, Inuiguchi and Sakawa [10] and Mausser and Laguna [11] propose to find a solution that will be close to optimal, regardless of the values eventually taken by the coefficients of the objective function. Hence, they approach the problem by using the minimax or the minimax relative regret criterion.

Mathematical programs with interval coefficients in the objective function as well as in the constraints have also been addressed in the literature. In this case, the focus is on the problem of computing the optimal value range, i.e., the range between the best and the worst optimal objective function values and the settings of the coefficients which provide the two extreme cases. These two extreme values allow the decision maker to better understand the risk involved and to gain an insight into the likelihood of these extremes [12]. For linear problems, Shaocheng [13] determines the linear programs which provide the best and worst possible optimal solutions when all variables are nonnegative variables and all constraints are inequalities. For the same problem, Chinneck and Ramadan [12] extend this study by comprehensively analyzing all kinds of variables and constraints and providing algorithms which allow them to determine the best and worst optimum and the coefficients which achieve these two extremes. Based on the properties of linear systems with interval coefficients, Hladík [14] proposes a unified approach for the problem of computing the optimal value range in linear problems with interval coefficients that allows for some dependences between coefficients. Very few results have been obtained for nonlinear problems. In fact, only nonlinearities in the objective function have been analyzed in [15] in linear fractional programs and in [16] in convex quadratic problems. Notice that mathematical programming with interval coefficients can be considered as an extension of classical sensitivity analysis as it deals with simultaneous and independent perturbations of the parameters.

This paper addresses linear bilevel problems whose objective function coefficients are assumed to lie between specific bounds. That is to say, f_1 and f_2 are linear functions with interval coefficients and the common constraint region S is a polyhedron. The purpose of this paper is to solve the optimal value range problem. To our knowledge, this is the first time that interval coefficients in bilevel programming have been considered. Notice that although the term linear is used in the description of these bilevel problems, the fact that the feasible region is implicitly determined by another mathematical program makes them nonlinear problems. This characteristic prevents the use of the properties of linear systems which form the base of the techniques used in previous works in mathematical programming with interval coefficients and makes the study of linear bilevel problems with interval coefficients more difficult. Focusing on the optimal value range problem, we will prove that the best and worst optimal solutions with respect to the upper level objective function occur at extreme points of the polyhedron S. Moreover, we will develop two enumerative algorithms that allow us to compute them as well as determining settings of the interval coefficients which provide these values.

The paper is organized as follows. Section 2 states the linear bilevel problem with interval coefficients (LBPIC). Section 3 goes on to analyze the LBPIC when the interval coefficients are only in the upper level objective function. Taking into account the properties of linear bilevel problems, the optimal value range can be obtained by solving two linear bilevel problems. In Section 4, the LBPIC is analyzed when the interval coefficients are only in the lower level objective function. In this case, the feasible region depends on the intervals, thus making the study more complex. Two algorithms are proposed to obtain the best and worst optimal solutions based loosely on the ideas of ranking extreme points of the Kth-best algorithm. In Section 5, the approaches developed when analyzing previous cases separately are integrated to obtain the optimal value range for the LBPIC. Finally, our conclusions are presented in Section 6.

2. LBPIC problem formulation

Linear bilevel programming has been studied very extensively in the literature (see [17,18,3] and the references therein). The linear bilevel problem can be formulated as:

$$\min_{x,y}\ cx + dy \quad \text{where } y \text{ solves} \tag{2a}$$

$$\min_{y}\ ay \tag{2b}$$

$$\text{s.t. } Ax + By \leqslant b \tag{2c}$$

$$x \geqslant 0,\ y \geqslant 0, \tag{2d}$$

where $A \in \mathbb{R}^{m\times n_1}$, $B \in \mathbb{R}^{m\times n_2}$ and c, d, a, b are vectors of conformal dimensions. Assuming that problem (2) has a solution, an important property is that there is an extreme point of the polyhedron S defined by constraints (2c) and (2d) which solves the problem [6].

The LBPIC assumes that the objective function coefficients are interval-valued, i.e.,

$$(c, d) \in \Phi = \left\{(c, d) \in \mathbb{R}^{n_1+n_2} : c_i \in [c_i^l, c_i^r],\ i = 1, \dots, n_1;\ d_j \in [d_j^l, d_j^r],\ j = 1, \dots, n_2\right\} \tag{3a}$$

$$a \in \Psi = \left\{a \in \mathbb{R}^{n_2} : a_j \in [a_j^l, a_j^r],\ \text{for } j = 1, \dots, n_2\right\}. \tag{3b}$$

Let S_1 be the projection of S onto $\mathbb{R}^{n_1}$. Let $a \in \Psi$. For given $x \in S_1$, the lower level decision maker solves the following linear problem:

$$\begin{aligned} LP_a(x): \quad & \min_y \quad ay \\ & \text{s.t.} \quad By \leqslant b - Ax \\ & \qquad\ \ y \geqslant 0. \end{aligned} \tag{4}$$

Let us denote by $S(x)$ its feasible region and by $M_a(x)$ the set of optimal solutions of problem (4). Then, the set

$$\mathrm{FR}_a = \{(x, y) : x \in S_1,\ y \in M_a(x)\} \tag{5}$$

is the feasible region of the associated bilevel problems obtained when $(c, d) \in \Phi$.

Since there are interval coefficients in the upper level objective function of the LBPIC, its objective function is interval valued. Moreover, its feasible region also depends on interval coefficients since it is implicitly defined by the lower level problem and there are interval coefficients in the lower level objective function. Clearly, different values of vectors (c, d) and a in their ranges produce different optimal upper level objective function values and different feasible regions, respectively. In this paper, we are interested in solving the optimal value range problem i.e., in computing the best and worst optimal upper level objective function values and determining the settings of the interval coefficients which provide these values.

In what follows, we assume that the polyhedron S is non-empty and bounded. Moreover, we denote by $LB(c, d, a)$ the linear bilevel problem obtained by setting the interval coefficients of the LBPIC at specific values in their ranges $(c, d) \in \Phi$ and $a \in \Psi$ and define $\mathcal{G}$, the set of optimal solutions of problems $LB(c, d, a)$:

$$\mathcal{G} = \{(x, y) : (x, y) \text{ solves the problem } LB(c, d, a),\ \text{ for some } (c, d) \in \Phi,\ a \in \Psi\}.$$

Definition 1. Let (x^*, y^*) be an optimal solution of the problem $LB(c^*, d^*, a^*)$. This point is the best optimal solution of the LBPIC if it provides the best value of the upper level objective function amongst the points of $\mathcal{G}$, that is to say,

$$c^*x^* + d^*y^* \leqslant cx + dy, \quad \forall (x, y) \in \mathcal{G}.$$

Similarly, (x^*, y^*) is the worst optimal solution if

$$cx + dy \leqslant c^*x^* + d^*y^*, \quad \forall (x, y) \in \mathcal{G}.$$

Remark 2. As each problem $LB(c, d, a)$ with c, d and a fixed within their ranges is a linear bilevel problem, there exists an extreme point of S which solves it. As a result, the best and worst optimal solutions of the LBPIC occur at vertices of S.

The following examples give us an insight into the optimal value range problem when dealing with bilevel problems and the difficulties which arise when solving it. They allow us to conclude that there is no precise way of systematizing the specific values of the interval coefficients that can be used to compute the best and worst possible optimal solutions of the LBPIC.

Example 1. The first example, in $\mathbb{R}^2$, involves the variable x controlled by the upper level decision maker and the variable y controlled by the lower level decision maker:

$$\begin{aligned} &\min_{x,y} \quad [1, 2]x + [-2, -1]y, \quad \text{where } y \text{ solves} \\ &\min_y \quad [-1, 2]y \\ &\text{s.t.} \quad (x, y) \in S \end{aligned}$$

where $S = \mathrm{conv}((2, 4), (3, 7), (9, 9), (12, 3), (5, 1))$ and conv stands for the convex hull. Set S is shown in gray in Fig. 1. The left part of Fig. 1 displays in bold FR_a when $a \in [-1, 0)$. The right part of the figure shows in bold FR_a when $a \in (0, 2]$. For $a = 0$, FR_0 is the whole polyhedron S.

The point $(3, 7)$ is the optimal solution of the LBPIC when $c \in [1, 2]$, $d \in [-2, -1]$, $a \in [-1, 0]$ and the point $(2, 4)$ is the optimal solution when $c \in [1, 2]$, $d \in [-2, -1]$, $a \in (0, 2]$. Therefore, $\mathcal{G} = \{(3, 7), (2, 4)\}$ and the best optimal solution is $(3, 7)$, which is obtained by setting $c = 1$, $d = -2$ and $a \in [-1, 0]$. The worst optimal solution is $(2, 4)$, which is obtained by setting $c = 2$, $d = -1$ and $a \in (0, 2]$. The optimal value range of the LBPIC is $[-11, 0]$.

Note that the point $(12, 3)$ provides the worst upper level objective function value $f_1 = 21$ and it is a feasible solution of all problems $LB(c, d, a)$, $a \in [-1, 2]$. However, it does not provide the worst optimal solution since that point is not an optimal solution for any problem $LB(c, d, a)$, $a \in [-1, 2]$.

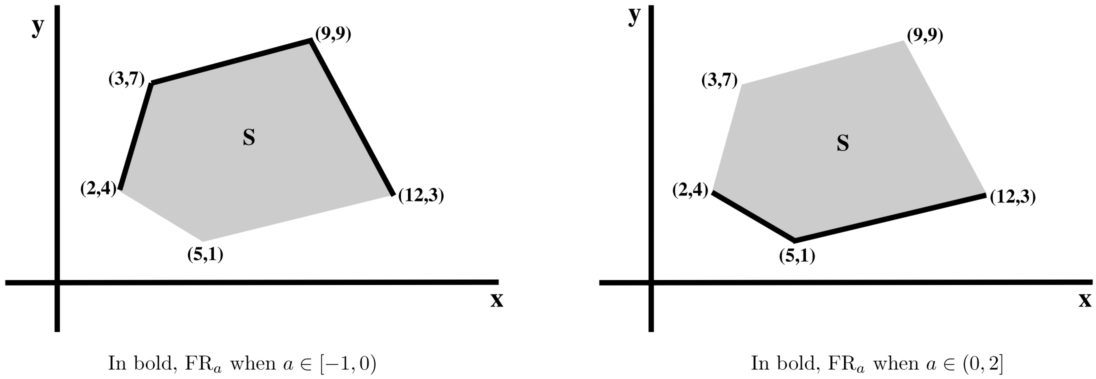

Fig. 1. Region S and feasible regions of Example 1. When $a = 0$, $\mathrm{FR}_0 = S$.

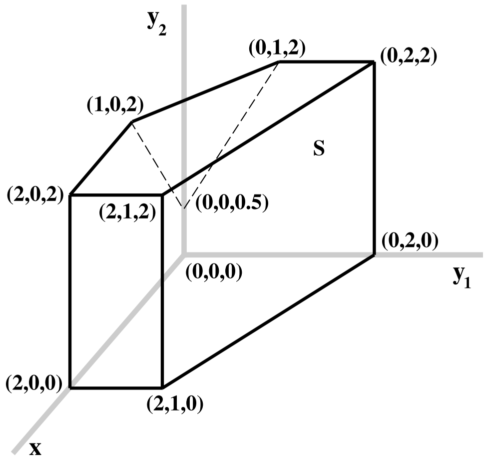

Fig. 2. Region S of Example 2.

Example 2.

$$\begin{array}{ll} \min\limits_{x,y_1,y_2} & x + y_1 + y_2, \quad \text{where } y = (y_1, y_2) \text{ solves} \\ \min\limits_{y_1,y_2} & [a_1^l, a_1^r]y_1 + [a_2^l, a_2^r]y_2 \\ \text{s.t.} & (x, y_1, y_2) \in S \end{array}$$

where $S = \{(x, y_1, y_2) : -3x - 3y_1 + 2y_2 \leqslant 1; x + 2y_1 \leqslant 4; y_2 \leqslant 2; x \leqslant 2; x, y_1, y_2 \geqslant 0\}$. The region S is shown in Fig. 2.

If $[a_1^l, a_1^r] = [-2, -1]$ and $[a_2^l, a_2^r] = [-3, -1]$, then $\mathrm{FR}_a = \mathrm{conv}((2, 1, 2), (0, 2, 2)),\ \forall a$. As a consequence, taking into account that the upper level objective function has no interval coefficients, the best and worst optimal solutions are the same and can be obtained by solving the bilevel problem for any value of a. The point $(0, 2, 2)$ is the optimal solution with $f_1 = 4$.

Similarly, if $[a_1^l, a_1^r] = [1, 2]$ and $[a_2^l, a_2^r] = [1, 3]$, then $\mathrm{FR}_a = \mathrm{conv}((2, 0, 0), (0, 0, 0)),\ \forall a$ and the optimal solution of the bilevel problem is $(0, 0, 0)$ with $f_1 = 0$.

In the previous cases, due to the fact that all problems $LB(c, d, a)$ have a common feasible region, the upper level decision maker needs to solve only one bilevel problem in order to obtain the best and the worst optimal solutions of the LBPIC. But this is not always the case. For instance, if $[a_1^l, a_1^r] = [-1, 2]$ and $[a_2^l, a_2^r] = [1, 3]$, then

$$\mathrm{FR}_a = \begin{cases} \mathrm{conv}((2, 0, 0), (0, 0, 0)), & \text{if } a_1 \in (0, 2],\ a_2 \in [1, 3] \\ \mathrm{conv}((2, 1, 0), (0, 2, 0)), & \text{if } a_1 \in [-1, 0),\ a_2 \in [1, 3] \\ \mathrm{conv}((2, 0, 0), (2, 1, 0), (0, 0, 0), (0, 2, 0)), & \text{if } a_1 = 0,\ a_2 \in [1, 3]. \end{cases}$$

In this case, $\mathcal{G} = \{(0, 0, 0), (0, 2, 0)\}$. The best optimal solution is $(0, 0, 0)$, which can be obtained by setting the interval coefficients, for instance, in the upper bounds $a_1 = 2$, $a_2 = 3$. The worst optimal solution is $(0, 2, 0)$, which can be obtained, for instance, by setting the interval coefficients to their lower bounds $a_1 = -1$, $a_2 = 1$.

To sum up, depending on the problem, we can obtain either the best optimal solution or the worst optimal solution setting the interval coefficients in their lower bounds, upper bounds or at some intermediate value. Hence, we can conclude that, in general, we cannot determine in advance which of the settings of the interval coefficients allow us to compute either the best optimal solution or the worst optimal solution.

In the following two sections, we will analyze separately the cases in which interval coefficients are either in the upper level objective function only or in the lower level objective function only. This analysis makes it easy to understand how the existence of interval coefficients in each objective function requires different approaches to deal with the optimal value range problem. Then, we will integrate both approaches to solve the LBPIC.

3. The LBPIC-1: interval coefficients in the upper level objective function only

The LBPIC-1 assumes that vectors c and d are interval valued but vector a is a fixed row vector of dimension n_2. Therefore, for a given $x \in S_1$, the lower level decision maker solves the following linear problem:

$$\begin{aligned} \min_y \quad & ay \\ \text{s.t.} \quad & y \in S(x). \end{aligned} \tag{6}$$

Let $M(x)$ be the set of optimal solutions of problem (6). Then, the set

$$\mathrm{FR} = \{(x, y) : x \in S_1,\ y \in M(x)\}$$

is the feasible region of problems $LB(c, d, a)$, for all $(c, d) \in \Phi$. That is to say, this feasible region is fixed whatever the settings of the interval coefficients of the LBPIC-1 in their range. As a result, we can consider that FR, which is implicitly defined, is the feasible region of the LBPIC-1. Hence, as proved in the following two theorems, the best and worst objective function values can be obtained by solving two linear bilevel problems in which the coefficients of the upper level objective function are fixed at their lower bound and their upper bound, respectively.

Theorem 3. *The best optimum of the LBPIC-1 is obtained by solving the problem $LB(c^l, d^l, a)$. Moreover, there exists an extreme point of S which solves the problem.*

Proof. Let (x^l, y^l) be an optimal solution of the problem $LB(c^l, d^l, a)$ and $(x, y) \in \mathcal{G}$. Then, (x, y) is an optimal solution of some problem $LB(c, d, a),\ (c, d) \in \Phi$. Since both problems have the same feasible region FR:

$$c^l x^l + d^l y^l \leqslant c^l x + d^l y.$$

Since $(c, d) \in \Phi$ and the variables are nonnegative:

$$c^l x + d^l y \leqslant cx + dy.$$

Hence, $c^l x^l + d^l y^l \leqslant cx + dy,\ \forall (x, y) \in \mathcal{G}$ and so (x^l, y^l) is the best optimal solution.

The second part of the theorem follows taking into account that $LB(c^l, d^l, a)$ is a linear bilevel problem. □

Theorem 4. *The worst optimum of the LBPIC-1 is obtained by solving the problem $LB(c^r, d^r, a)$. Moreover, there exists an extreme point of S which solves the problem.*

Proof. Similar to that of Theorem 3. □

4. The LBPIC-2: interval coefficients in the lower level objective function only

The LBPIC-2 assumes that vector (c, d) is a fixed row vector of dimension $n_1 + n_2$ and vector a is interval valued. In this case, the existence of interval coefficients in the lower level objective function affects the definition of the feasible region of the bilevel problem. In fact, this feasible region changes as the coefficients of the lower level objective function change.

Assuming conditions on the feasible regions of the linear bilevel problems defined by the extreme points of Ψ, the following theorem proves that the best and worst optimal solutions can be obtained by solving a linear bilevel problem.

Theorem 5. *Let E_Ψ be the set of extreme points of Ψ. If FR_a is the same for all $a \in E_\Psi$, then the best and worst optimal solutions of the problem LBPIC-2 coincide and can be obtained by solving the linear bilevel problem:*

$$\begin{aligned} \min_{x,y} \quad & cx + dy, \quad \textit{where } y \textit{ solves} \\ \min_y \quad & a^l y \\ \textit{s.t.} \quad & (x, y) \in S. \end{aligned} \tag{7}$$

Proof. Let $\tilde{a} \in \Psi$. Since Ψ is a compact polyhedron, then $\tilde{a} = \sum_{j=1}^{k} \lambda_j p_j$, where $p_j \in E_\Psi,\ j = 1, \ldots, k,\ \lambda_j \in \mathbb{R},\ \lambda_j \geqslant 0,\ j = 1, \ldots, k$ and $\sum_{j=1}^{k} \lambda_j = 1$.

Let FR_E be the feasible region of problems $LB(c, d, a),\ a \in E_\Psi$, i.e. for all $a \in E_\Psi$, $\mathrm{FR}_a = \mathrm{FR}_E$. Let $(\hat{x}, \hat{y}) \in \mathrm{FR}_E$. Then, $\hat{y}$ is an optimal solution of problem $LP_a(\hat{x})$ defined in (4), for all $a \in E_\Psi$. So, $\hat{y}$ is an optimal solution of problem $LP_{p_j}(\hat{x}),\ j = 1, \ldots, k$.

Note that problems $LP_a(\hat{x}),\ a \in \Psi$, all have the same feasible region $S(\hat{x})$. As a consequence, $\hat{y}$ is a feasible solution of problem $LP_{\tilde{a}}(\hat{x})$. Moreover,

$$\tilde{a}\hat{y} = \sum_{j=1}^{k} \lambda_j p_j \hat{y} \leqslant \sum_{j=1}^{k} \lambda_j p_j y = \tilde{a}y, \quad \forall y \in S(\hat{x}).$$

Hence, $\hat{y}$ is an optimal solution of problem $LP_{\tilde{a}}(\hat{x})$, $(\hat{x}, \hat{y}) \in \mathrm{FR}_{\tilde{a}}$ and $\mathrm{FR}_E \subseteq \mathrm{FR}_{\tilde{a}}$.

Assume now that $(\hat{x}, \hat{y}) \in \mathrm{FR}_{\tilde{a}}$. Then, $\hat{y}$ is an optimal solution of problem $LP_{\tilde{a}}(\hat{x})$. If $(\hat{x}, \hat{y}) \notin \mathrm{FR}_E$, then $\hat{y}$ is not an optimal solution of problem $LP_a(\hat{x}),\ a \in E_\Psi$. Hence, there exists $\bar{y}$ a feasible solution of problems $LP_a(\hat{x}),\ a \in E_\Psi$, so that $a\bar{y} < a\hat{y},\ a \in E_\Psi$. In particular, $p_j\bar{y} < p_j\hat{y},\ j = 1, \ldots, k$. Then,

$$\tilde{a}\bar{y} = \sum_{j=1}^{k} \lambda_j p_j \bar{y} < \sum_{j=1}^{k} \lambda_j p_j \hat{y} = \tilde{a}\hat{y},$$

in contradiction, as $\bar{y}$ is also a feasible solution of problem $LP_{\tilde{a}}(\hat{x})$. This implies that $\mathrm{FR}_{\tilde{a}} \subseteq \mathrm{FR}_E$.

Hence, we can consider FR_E as the feasible region of the LBPIC-2. Since the upper level objective function has no interval coefficients, the LBPIC-2 reduces to a classic linear bilevel problem in which FR_E can be obtained either by using the lower bound of the interval coefficients, as indicated in the theorem, or by using any other $a \in \Psi$. This completes the proof. □

It is worth mentioning that to check the conditions under which Theorem 5 holds is not easy because the feasible region of a bilevel problem is implicitly defined. What is known from the literature in linear bilevel programming is that $\mathrm{FR}_a,\ a \in \Psi$ is formed by the union of faces of the polyhedron S, but there is not an explicit expression that tells us which these faces are.

Apart from the special case proved in the previous theorem, the examples in Section 2 have shown that there is no general expression that we can use in advance to solve the optimal value range problem when there are interval coefficients in the lower level objective function. However, as we will prove below, the properties of linear bilevel problems allow us to develop algorithms which compute the best and worst optimal solutions of the LBPIC-2 and the settings of the interval coefficients which provide them, without explicitly writing the expression of the feasible regions.

For this purpose, we make use of the fact that each problem $LB(c, d, a)$ has at least one extreme point of S which is an optimal solution. The general idea of the algorithms proposed is similar to that used in [17] to develop the Kth-best algorithm for solving linear bilevel problems. This algorithm orders the extreme points of the region S by ascending values of the upper level objective function. Then, the first of these extreme points which is a feasible point is an optimal solution of the bilevel problem. The essence of the Kth-best algorithm is that the ordering can be made sequentially by computing successively adjacent extreme points to the incumbent extreme point.

4.1. KBB: an algorithm for finding the best optimal solution

Let us consider the relaxed problem of the LBPIC-2 obtained by ignoring the lower level objective function:

$$\begin{aligned} \min_{x,y} \quad & cx + dy \\ \text{s.t.} \quad & (x, y) \in S. \end{aligned} \tag{8}$$

This is a linear problem whose optimal solution (x^*, y^*) provides a lower bound of the best optimum of the LBPIC-2. Clearly, if there exists $a^* \in \Psi$ such that $(x^*, y^*) \in \mathrm{FR}_{a^*}$, then (x^*, y^*) is the best optimal solution of the LBPIC-2. This is the case, for instance, if $0 \in [a_j^l, a_j^r],\ \forall j = 1, \ldots, n_2$ as then $\mathrm{FR}_0 = S$.

In general, in order to determine if (x^*, y^*) is the best optimal solution of the LBPIC-2, we should find $a^* \in \Psi$, such that y^* is an optimal solution of the linear problem $LP_{a^*}(x^*)$. For this purpose, denoting by u the vector of variables of the dual problem of (4) and taking into account duality properties in linear programming, it suffices to find a solution of the following linear system:

$$\begin{aligned} & -u^t B \leqslant a \\ & ay^* = (Ax^* - b)^t u \\ & a_j^l \leqslant a_j \leqslant a_j^r, \quad j = 1, \ldots, n_2 \\ & u \geqslant 0 \end{aligned} \tag{9}$$

where t stands for transpose.

As we will prove, the best extreme point of S with respect to the upper level objective function which is a feasible solution of the LBPIC-2 for some $a \in \Psi$ will be an optimal solution of the corresponding problem $LB(c, d, a)$ and thus it will be the best optimal solution of the LBPIC-2. In order to compute that extreme point, we can use the same general idea of the Kth-best avoiding having to compute all extreme points of S in advance. Instead, the ordered best extreme points of S with respect to the upper level objective function are sequentially computed until a point of FR_a for some $a \in \Psi$ is identified. This is the best optimal solution. Below is the description of the algorithm in which $W^{[i]}$ denotes the set of adjacent extreme points of $(x^{[i]}, y^{[i]})$ and T denotes the set of analyzed and discarded extreme points.

Step 1.
Let $(x^{[1]}, y^{[1]})$ be an optimal solution to problem (8).
Set $W = \{(x^{[1]}, y^{[1]})\}$ and $T = \emptyset$. Set $i = 1$.
Step 2.
Set $(x^*, y^*) = (x^{[i]}, y^{[i]})$ and check system (9).
If the system is feasible, stop: $(x^{[i]}, y^{[i]})$ is the best optimal solution.
Step 3.
Set $T = T \cup \{(x^{[i]}, y^{[i]})\}$ and $W = (W \cup W^{[i]}) \setminus T$.
Step 4.
Set $i = i + 1$ and choose $(x^{[i]}, y^{[i]})$ so that
$cx^{[i]} + dy^{[i]} = \min\{cx + dy : (x, y) \in W\}$.
Go to Step 2.

Next, we describe in more detail, the steps of the algorithm. First, an extreme point of S which is an optimal solution to the relaxed problem (8) is computed. For the incumbent extreme point, in Step 2 it is tested if there exists $a^* \in \Psi$ so that the extreme point is a point of FR_{a^*}. For this purpose, system (9) is checked for feasibility. If it is feasible, we conclude that the incumbent extreme point has to be the best optimal solution of the LBPIC-2. This will be proved in Theorem 6. If system (9) is not feasible, this extreme point can be discarded. Then, the set of adjacent extreme points to the incumbent extreme point is computed and added to the set W of extreme points waiting to be analyzed. After eliminating the extreme points already discarded, an extreme point of W with the best value of the upper level objective function is selected to be the incumbent extreme point and we repeat the process. Upon termination, as proved in Theorem 7, an extreme point of S with the best value of the upper level objective function is provided which is a point of $\mathrm{FR}_{\tilde{a}}$, for some $\tilde{a} \in \Psi$ computed by the algorithm.

Theorem 6. *Let $(x^*, y^*) = (x^{[k]}, y^{[k]})$ be the best extreme point of S, according to the upper level objective function, so that there exists $a^* \in \Psi$ such that $(x^*, y^*) \in \mathrm{FR}_{a^*}$. Then, the point (x^*, y^*) is the best optimal solution of the LBPIC-2 and the vector $a^* \in \Psi$ gives the settings of the interval coefficients which provide it.*

Proof. Let us consider the problem $LB(c, d, a^*)$. Taking into account the properties of linear bilevel problems, there exists an extreme point of S which solves the problem. Assume that (x^*, y^*) was not an optimal solution of this problem. Hence, there would exist $(\tilde{x}, \tilde{y})$ extreme point of S such that $(\tilde{x}, \tilde{y}) \in \mathrm{FR}_{a^*}$ and

$$c\tilde{x} + d\tilde{y} < cx^* + dy^*.$$

Then $(\tilde{x}, \tilde{y}) = (x^{[i]}, y^{[i]}),\ i \in \{1, \ldots, k-1\}$, which contradicts the hypothesis. Therefore, (x^*, y^*) is an optimal solution of the problem $LB(c, d, a^*)$ and so $(x^*, y^*) \in \mathcal{G}$.

Moreover, if $(x, y) \in \mathcal{G}$, then there exists $a \in \Psi$ so that $(x, y) \in \mathrm{FR}_a$. Hence, it follows from the hypothesis that

$$cx^* + dy^* \leqslant cx + dy$$

and so (x^*, y^*) is the best optimal solution of the LBPIC-2. □

Theorem 7. *The algorithm KBB finds the best optimal solution of the LBPIC-2.*

Proof. It suffices to take into account that the number of extreme points of S is finite and that the $(k+1)$st best extreme point according to the upper level objective function $(x^{[k+1]}, y^{[k+1]})$ is adjacent to $(x^{[1]}, y^{[1]})$ or $(x^{[2]}, y^{[2]})$, ... or $(x^{[k]}, y^{[k]})$ (see [5]).

Moreover, if the kth best extreme point is the first one to be a point of $\mathrm{FR}_{\tilde{a}}$, for some $\tilde{a} \in \Psi$ then, by applying Theorem 6, this extreme point solves the corresponding $LB(c, d, \tilde{a})$ and so it is the best optimal solution. Moreover, $\tilde{a}$ are the settings of the interval coefficients of the lower level objective function which provide the best optimum. □

4.2. KBW: an algorithm for finding the worst optimal solution

This problem, although close to the previous one, turns out to be much harder to solve. The general idea of the algorithm is very similar, that is to say, we sort the extreme points of S in descending order of the upper level objective function value. In order to select the worst optimal solution of the LBPIC-2, we should select the first extreme point which solves the $LB(c, d, \tilde{a})$ for some $\tilde{a} \in \Psi$. However, unlike the previous case, now the fact that the incumbent extreme point is a point of $\mathrm{FR}_{\tilde{a}}$, for some $\tilde{a} \in \Psi$ does not guarantee that it is an optimal solution of the corresponding $LB(c, d, \tilde{a})$ (see Example 1). Hence, both feasibility and optimality conditions have to be tested in the algorithm. As in the KBB algorithm, the extreme points of S do not need to be computed and ordered a priori, but the sorting is made sequentially computing each time only adjacent extreme points to the incumbent one.

Let (x^*, y^*) be the incumbent extreme point. In order to check its feasibility, system (9) should be solved. If it is not feasible, we can conclude that the incumbent point is not in $\mathrm{FR}_a,\ a \in \Psi$. In contrast, if the system is feasible, there exists a set $\Psi_{(x^*,y^*)} \subseteq \Psi$ so that the incumbent extreme point is a feasible point of the corresponding $LB(c, d, a),\ a \in \Psi_{(x^*,y^*)}$. To check if the point (x^*, y^*) is the worst optimal solution of the LBPIC-2, we need to find $a^* \in \Psi_{(x^*,y^*)}$ so that (x^*, y^*) solves $LB(c, d, a^*)$. In order to compute $\Psi_{(x^*,y^*)}$, we propose to use the Karush–Kuhn–Tucker conditions [19] and determine the vectors $a \in \Psi$ which can be represented as a nonnegative linear combination of the gradients of the binding constraints at (x^*, y^*):

$$\Psi_{(x^*,y^*)} = \{a \in \Psi : uG = a,\ u \geqslant 0\} \tag{10}$$

where $Gy = g$ are the set of inequalities from $-By \geqslant -(b - Ax),\ y \geqslant 0$ that are binding at (x^*, y^*) and u is a row vector of conformal dimension. After computing $\Psi_{(x^*,y^*)}$ and taking into account the linearity of the upper level objective function, to look for $a^* \in \Psi_{(x^*,y^*)}$, we propose an evaluation of the adjacent extreme points of (x^*, y^*).

Below is the description of the algorithm in which $W^{[i]}$ denotes the set of adjacent extreme points of $(x^{[i]}, y^{[i]})$, T denotes the set of analyzed and discarded extreme points and W^e denotes the set of discarded extreme points because they are never feasible solutions.

Step 1.
Let $(x^{[1]}, y^{[1]})$ be an optimal solution of the problem
$$\begin{array}{ll} \max_{x,y} & cx + dy \\ \text{s.t.} & (x, y) \in S. \end{array}$$
Let $W = \{(x^{[1]}, y^{[1]})\}$, $W^e = \emptyset$, $W^p = \emptyset$ and $T = \emptyset$. Set $i = 1$.

Step 2.
Compute $\Psi_{(x^{[i]},y^{[i]})}$. If $\Psi_{(x^{[i]},y^{[i]})} = \emptyset$, set $W^e = W^e \cup \{(x^{[i]}, y^{[i]})\}$ and go to Step 8.
Set $W^p = W^{[i]} \setminus (T \cup W^e)$. If $W^p = \emptyset$, stop, $(x^{[i]}, y^{[i]})$ is the worst optimal solution.
Set $\Psi^* = \Psi_{(x^{[i]},y^{[i]})}$.

Step 3.
If $W^p = \emptyset$, go to Step 8.

Step 4.
Select $(\tilde{x}, \tilde{y})$ so that $c\tilde{x} + d\tilde{y} = \max\{cx + dy : (x, y) \in W^p\}$.
Compute $\Psi_{(\tilde{x},\tilde{y})}$. If $\Psi_{(\tilde{x},\tilde{y})} = \emptyset$, set $W^e = W^e \cup \{(\tilde{x}, \tilde{y})\}$, $W^p = W^p \setminus \{(\tilde{x}, \tilde{y})\}$.
Go to Step 3.

Step 5.
If $\Psi^* \cap \Psi_{(\tilde{x},\tilde{y})} = \emptyset$, set $W^p = W^p \setminus \{(\tilde{x}, \tilde{y})\}$ and go to Step 3.

Step 6.
Set $\Psi^* = \Psi^* \setminus \Psi_{(\tilde{x},\tilde{y})}$. If $\Psi^* = \emptyset$, set $T = T \cup \{(x^{[i]}, y^{[i]})\}$ and go to Step 8.

Step 7.
Set $W^p = W^p \setminus \{(\tilde{x}, \tilde{y})\}$. If $W^p = \emptyset$, stop, $(x^{[i]}, y^{[i]})$ is the worst optimal solution.
Otherwise, go to Step 4.

Step 8.
Set $W = (W \cup W^{[i]}) \setminus (T \cup W^e)$. Set $i = i + 1$ and select $(x^{[i]}, y^{[i]})$ so that
$cx^{[i]} + dy^{[i]} = \max\{cx + dy : (x, y) \in W\}$.
Go to Step 2.

In Step 1, the extreme point with the worst value of the upper level objective function in the region S is selected. In Step 2, the subset of Ψ is determined for which the incumbent extreme point is a feasible solution of the associated problem $LB(c, d, a)$. In case this set is the empty set, the incumbent point is of no interest and can be discarded. Then the process goes on by computing in Step 8 the set of its adjacent extreme points and adding them to the set W of extreme points waiting to be analyzed. After eliminating the extreme points already discarded, the extreme point of this set with the maximum value of the upper level objective function, that is to say the worst value, is selected as the following extreme point to be analyzed.

Otherwise, the incumbent extreme point is a feasible solution of problems $LB(c, d, a),\ a \in \Psi^*$. Then the idea is to examine W^p, the set of all its adjacent extreme points of interest, looking for confirmation of the incumbent extreme point being the worst optimal solution or not. The adjacent extreme points of interest are those not previously examined or discarded. Hence, they have to be looked for among the extreme points with upper level objective function values not worse than the incumbent extreme point. Obviously, if there are no adjacent extreme points to be examined, we conclude that the incumbent extreme point is the worst optimal solution of the problem LBPIC-2.

Steps 3–7 allow us to check if the incumbent extreme point is an optimal solution of some of the bilevel problems for which it is a feasible solution. For this purpose, the extreme points of W^p are assessed in descending order of the upper level objective function value, trying to find a set of values $a \in \Psi^*$ for which the incumbent extreme point is an optimal solution of the corresponding bilevel problem, and so finishing the algorithm with the worst optimal solution of the LBPIC-2. If this is not possible, the process continues in Step 8 as indicated above.

Specifically, in Step 4, the worst point in W^p is selected and is discarded if it is not a feasible solution of $LB(c, d, a),\ a \in \Psi$. Otherwise, in Step 5, we test if the set of points $a \in \Psi$ for which it is a feasible solution of the associated linear bilevel problem, has points in common with Ψ^*. In the negative case, this point is rejected and the process goes on to Step 3. If it has points in common, for the common points the incumbent extreme point cannot be an optimal solution of the associated linear bilevel problem since the examined extreme point is not worse than the incumbent extreme point. Hence, the set of common points is eliminated from Ψ^*. If the new Ψ^* has no points, then the incumbent extreme point will never be an optimal solution of the linear bilevel problems for which it is feasible and so its analysis is finished. The process continues in Step 8 as indicated above. Otherwise, the analysis goes on either stopping the algorithm if there are no points in W^p or selecting a new point of W^p in Step 4 to be examined. Upon termination, an extreme point of S with the worst value of the upper level objective function is provided which solves the problem $LB(c, d, a^*)$, for some $a^* \in \Psi^*$, as we prove in the following theorem.

Theorem 8. *The algorithm KBW finds the worst optimal solution of the LBPIC*-2.

Proof. It suffices to take into account that the number of extreme points of S is finite and that the $(k + 1)$st worst extreme point according to the upper level objective function $(x^{[k+1]}, y^{[k+1]})$ is adjacent to $(x^{[1]}, y^{[1]})$ or $(x^{[2]}, y^{[2]}), \ldots$ or $(x^{[k]}, y^{[k]})$.

Moreover, the algorithm always finishes with the worst extreme point that solves the $LB(c, d, a^*)$ for some $a^* \in \Psi^*$ and so it is the worst optimal solution. Finally, a^* are the settings of the interval coefficients of the lower level objective function which provide the worst optimum. □

5. The LBPIC

Having analyzed separately the LBPIC-1 and LBPIC-2, we are now in a position to integrate the results and deal with the LBPIC as a whole. On the one hand, the existence of interval coefficients in the lower level objective function causes, in general, the feasible regions of problems $LB(c, d, a), (c, d) \in \Phi,\ a \in \Psi$, to be different. Nevertheless, as they are formed by the union of faces of the polyhedron S, the number of different feasible regions which can appear when varying $a \in \Psi$ is finite.

On the other hand, it is clear that in order to obtain the best optimal solution of the LBPIC, we should select the coefficients of the upper level objective function at their most favorable form. Similarly, to obtain the worst optimal solution, we should select the coefficients at their least favorable form. These remarks allow us to conclude the following theorems.

Theorem 9. *Let E_Ψ be the set of extreme points of Ψ. If FR_a is the same for all $a \in E_\Psi$, then the best optimal solution of the problem LBPIC can be obtained by solving the linear bilevel problem:*

$$\begin{array}{ll} \min\limits_{x,y} & c^l x + d^l y, \quad \textit{where } y \textit{ solves} \\ \min\limits_{y} & a^l y \\ s.t. & (x, y) \in S. \end{array} \tag{11}$$

Similarly, the worst optimal solution of the problem LBPIC can be obtained by solving the linear bilevel problem:

$$\begin{array}{ll} \min\limits_{x,y} & c^r x + d^r y, \quad \textit{where } y \textit{ solves} \\ \min\limits_{y} & a^l y \\ s.t. & (x, y) \in S. \end{array} \tag{12}$$

Proof. Let FR_E be the feasible region of problems $LB(c, d, a), (c, d) \in \Phi,\ a \in E_\Psi$. As in the proof of Theorem 5, under the hypothesis we can conclude that FR_E is the feasible region for all problems $LB(c, d, a),\ (c, d) \in \Phi,\ a \in \Psi$, and so it can be considered as the feasible region of the LBPIC. Hence, LBPIC reduces to an LBPIC-1 with FR_E as the feasible region and the conclusion follows from Theorems 3 and 4. □

Theorem 10. *The best optimal solution of the LBPIC is obtained by applying the KBB algorithm to the LBPIC*-2 *with the upper level objective function $f_1(x, y) = c^l x + d^l y$.*

The worst optimal solution of the LBPIC is obtained by applying the KBW algorithm to the problem LBPIC-2 *with the upper level objective function $f_1(x, y) = c^r x + d^r y$.*

Proof. Let $(x^*, y^*) \in \mathcal{G}$ be the best optimal solution. Then, there exist $(c^*, d^*) \in \Phi$, $a^* \in \Psi$ so that (x^*, y^*) is an optimal solution of the problem $LB(c^*, d^*, a^*)$ and

$$c^* x^* + d^* y^* \leqslant cx + dy, \quad \forall (x, y) \in \mathcal{G}$$

where (x, y) solves the problem $LB(c, d, a), (c, d) \in \Phi,\ a \in \Psi$.

Let us consider the problem $LB(c^l, d^l, a^*)$. Note that this problem and the problem $LB(c^*, d^*, a^*)$ have the same feasible region. Let (x^l, y^l) be an optimal solution of the problem $LB(c^l, d^l, a^*)$. Hence $(x^l, y^l) \in \mathcal{G}$ and

$$c^l x^l + d^l y^l \leqslant c^l x^* + d^l y^* \leqslant c^* x^* + d^* y^*.$$

In order to avoid a contradiction, all previous inequalities are equalities and (x^l, y^l) is the best optimal solution. Hence, in order to find the best optimal solution we should apply the KBB algorithm to the LBPIC-2 with the upper level objective function $f_1(x, y) = c^l x + d^l y$.

The proof of the second part of the theorem is similar. □

In Appendix, an example is given to exhaustively show all the steps in the procedure.

Concerning the complexity of the problem, the LBPIC is NP-hard, as it includes the linear bilevel problem as a particular case. Note that constants can be represented by degenerated intervals where the lower and upper bounds are equal.

Remark 11. The best optimal value of the LBPIC can also be obtained as a solution of the following quadratic bilevel problem in which the coefficients in both objective functions are regarded as additional optimization variables controlled by the upper level decision maker:

$$\begin{array}{ll} \min\limits_{x,c,d,a,y} & cx + dy \\ \text{s.t.} & (c, d) \in \Phi,\ a \in \Psi,\ \text{where } y \text{ solves} \\ & \min\limits_{y} ay \\ & \text{s.t.} (x, y) \in S. \end{array} \tag{13}$$

Similarly to Theorem 10, we can conclude that the best optimal value of the LBPIC is obtained by solving:

$$\begin{array}{ll} \min\limits_{x,a,y} & c^l x + d^l y \\ \text{s.t.} & a \in \Psi,\ \text{where } y \text{ solves} \\ & \min\limits_{y} ay \\ & \text{s.t. } (x, y) \in S \end{array} \tag{14}$$

which is a linear bilinear bilevel problem, i.e. a particular case of a linear quadratic bilevel problem.

Convex quadratic bilevel problems have been addressed in [20–22]. In [2,3] and the references therein and [23], the linear quadratic bilevel problem is analyzed. Moreover, problem (14) is a particular case of the quasiconcave quadratic bilevel problem considered in [24]. However, we think that approaching the problem of computing the best optimal solution of the LBPIC by using the techniques proposed in these papers will possibly not be very helpful as it ignores the structure and properties of the linear bilevel problems involved.

On the other hand, it is worth mentioning that the algorithm KBB can be extended to cover the more general case in which the coefficients a lie on a bounded polyhedron. We should only modify system (9) changing the conditions on the parameter a.

6. Conclusions

In this paper, we have analyzed linear bilevel problems with interval coefficients in both objective functions. The aim was to solve the optimal value range problem, that is to say to provide the best and the worst upper level objective function values and the settings of the interval coefficients which provide them. Taking into account the differences in problems with interval coefficients in the upper level objective function only or in the lower level objective function only, both cases have been studied separately. The first can be approached by solving two linear bilevel problems. The complexity of the second one requires a more detailed analysis. We have proved by examples that there is no precise way of systematizing the specific values of the interval coefficients that can be used to compute the best and the worst possible optimal solutions. Hence, several particular cases which lead to easier solution procedures were considered first. Next, for the general case, two algorithms were proposed to deal with the problem of obtaining the best and worst optimal solutions. These algorithms are based on ranking extreme points and seeing if they can be an optimal solution for some of the linear bilevel problems obtained by setting the interval coefficients at values in their range. All the procedures are then integrated to solve the optimal value range problem for linear bilevel problems with interval coefficients in both the upper and lower objective functions.

Acknowledgments

The authors gratefully acknowledge the anonymous referees for their valuable suggestions to improve the presentation of the paper.

Appendix

To illustrate all the steps in the procedure of solving the optimal value range, we consider the region S of Example 2 with the upper level objective function $f_1 = [1, 4]x + [1, 3]y_1 + [1, 2]y_2$ and the lower level objective function $f_2 = [2, 5]y_1 + [-3, -1]y_2$. Thus, $\Phi = \{(c, d_1, d_2) : c \in [1, 4], d_1 \in [1, 3], d_2 \in [1, 2]\}$, $\Psi = \{(a_1, a_2) : a_1 \in [2, 5], a_2 \in [-3, -1]\}$.

In order to find the best optimal solution, we apply the KBB algorithm with the upper level objective function $f_1^l = x + y_1 + y_2$:

Step 1. $(x^{[1]}, y_1^{[1]}, y_2^{[1]}) = (0, 0, 0)$. Set $W = \{(0, 0, 0)\}$ and $T = \emptyset$. Set $i = 1$.
Step 2. $(x^*, y_1^*, y_2^*) = (0, 0, 0)$. System (9) is not feasible.
Step 3. $T = T \cup \{(0, 0, 0)\} = \{(0, 0, 0)\}$, $W = (W \cup W^{[1]}) \setminus T = \{(2, 0, 0), (0, 0, 0.5), (0, 2, 0)\}$.
Step 4. $(x^{[2]}, y_1^{[2]}, y_2^{[2]}) = (0, 0, 0.5)$.
Step 2. $(x^*, y_1^*, y_2^*) = (0, 0, 0.5)$. System (9) is feasible with $a_1 = 2$ and $a_2 = -1$.
The best optimal solution is $(0, 0, 0.5), f_1^l = 0.5$.

In order to compute the worst optimal solution, we apply the KBW algorithm with the upper level objective function $f_1^r = 4x + 3y_1 + 2y_2$.

Step 1. $(x^{[1]}, y_1^{[1]}, y_2^{[1]}) = (2, 1, 2), f_1^r(2, 1, 2) = 15$. $W = \{(2, 1, 2)\}$, $W^e = \emptyset$, $W^p = \emptyset$, $T = \emptyset$.
Step 2. $\Psi_{(2,1,2)} = \emptyset$. $W^e = \{(2, 1, 2)\}$.
Step 8. $W = (W \cup W^{[1]}) \setminus (T \cup W^e) = \{(2, 0, 2),\ (2, 1, 0),\ (0, 2, 2)\}$. $(x^{[2]}, y_1^{[2]}, y_2^{[2]}) = (2, 0, 2)$.
Step 2. $\Psi_{(2,0,2)} = \Psi \neq \emptyset$. $W^p = W^{[2]} \setminus (T \cup W^e) = \{(2, 0, 0), (1, 0, 2)\} \neq \emptyset$. $\Psi^* = \Psi_{(2,0,2)}$.
Step 3. $W^p \neq \emptyset$.
Step 4. $(\tilde{x}, \tilde{y}_1, \tilde{y}_2) = (2, 0, 0)$. $\Psi_{(2,0,0)} = \emptyset$, $W^e = \{(2, 1, 2)\} \cup \{(2, 0, 0)\}$, $W^p = \{(1, 0, 2)\}$.
Step 3. $W^p \neq \emptyset$.
Step 4. $(\tilde{x}, \tilde{y}_1, \tilde{y}_2) = (1, 0, 2)$. $\Psi_{(1,0,2)} = \Psi \neq \emptyset$.
Step 5. $\Psi^* \cap \Psi_{(1,0,2)} = \Psi = \Psi^* \neq \emptyset$.
Step 6. $\Psi^* = \Psi^* \setminus \Psi_{(1,0,2)} = \emptyset$. $T = \{(2, 0, 2)\}$.
Step 8. $W = (W \cup W^{[2]}) \setminus (T \cup W^e) = \{(2, 1, 0), (0, 2, 2), (1, 0, 2)\}$. $(x^{[3]}, y_1^{[3]}, y_2^{[3]}) = (2, 1, 0)$.
Step 2. $\Psi_{(2,1,0)} = \emptyset$. $W^e = \{(2, 1, 2), (2, 0, 0), (2, 1, 0)\}$.
Step 8. $W = (W \cup W^{[3]}) \setminus (T \cup W^e) = \{(0, 2, 2), (1, 0, 2), (0, 2, 0)\}$. $(x^{[4]}, y_1^{[4]}, y_2^{[4]}) = (0, 2, 2)$.
Step 2. $\Psi_{(0,2,2)} = \emptyset$. $W^e = \{(2, 1, 2), (2, 0, 0), (2, 1, 0), (0, 2, 2)\}$.
Step 8. $W = (W \cup W^{[4]}) \setminus (T \cup W^e) = \{(1, 0, 2), (0, 1, 2), (0, 2, 0)\}$. $(x^{[5]}, y_1^{[5]}, y_2^{[5]}) = (1, 0, 2)$.
Step 2. $\Psi_{(1,0,2)} = \Psi \neq \emptyset$. $W^p = W^{[5]} \setminus (T \cup W^e) = \{(0, 1, 2), (0, 0, 0.5)\} \neq \emptyset$. $\Psi^* = \Psi_{(1,0,2)}$.
Step 3. $W^p \neq \emptyset$.
Step 4. $(\tilde{x}, \tilde{y}_1, \tilde{y}_2) = (0, 1, 2)$. $\Psi_{(0,1,2)} = \{(a_1, a_2) : a_1 \in [2, 5], a_2 \in [-3, -1], 2a_1 + 3a_2 \leqslant 0\} \neq \emptyset$.
Step 5. $\Psi^* \cap \Psi_{(0,1,2)} = \Psi_{(0,1,2)} \neq \emptyset$.
Step 6. $\Psi^* = \Psi^* \setminus \Psi_{(0,1,2)} = \{(a_1, a_2) : a_1 \in [2, 5], a_2 \in [-3, -1], 2a_1 + 3a_2 > 0\} \neq \emptyset$.
Step 7. $W^p = W^p \setminus \{(0, 1, 2)\} = \{(0, 0, 0.5)\} \neq \emptyset$.
Step 4. $(\tilde{x}, \tilde{y}_1, \tilde{y}_2) = (0, 0, 0.5)$. $\Psi_{(0,0,0.5)} = \{(a_1, a_2) : a_1 \in [2, 5], a_2 \in [-3, -1], 2a_1 + 3a_2 \geqslant 0\} \neq \emptyset$.
Step 5. $\Psi^* \cap \Psi_{(0,0,0.5)} = \Psi^* \neq \emptyset$.
Step 6. $\Psi^* = \Psi^* \setminus \Psi_{(0,0,0.5)} = \emptyset$. $T = \{(2, 0, 2), (1, 0, 2)\}$.
Step 8. $W = (W \cup W^{[5]}) \setminus (T \cup W^e) = \{(0, 1, 2), (0, 2, 0), (0, 0, 0.5)\}$. $(x^{[6]}, y_1^{[6]}, y_2^{[6]}) = (0, 1, 2)$.
Step 2. $\Psi_{(0,1,2)} \neq \emptyset$. $W^p = W^{[6]} \setminus (T \cup W^e) = \{(0, 0, 0.5)\} \neq \emptyset$. $\Psi^* = \Psi_{(0,1,2)}$.
Step 3. $W^p \neq \emptyset$.
Step 4. $(\tilde{x}, \tilde{y}_1, \tilde{y}_2) = (0, 0, 0.5)$. $\Psi_{(0,0,0.5)} \neq \emptyset$.
Step 5. $\Psi^* \cap \Psi_{(0,0,0.5)} = \{(a_1, a_2) : a_1 \in [2, 5], a_2 \in [-3, -1], 2a_1 + 3a_2 = 0\} \neq \emptyset$.
Step 6. $\Psi^* = \Psi^* \setminus \Psi_{(0,0,0.5)} = \{(a_1, a_2) : a_1 \in [2, 5], a_2 \in [-3, -1], 2a_1 + 3a_2 < 0\} \neq \emptyset$.
Step 7. $W^p = W^p \setminus \{(0, 0, 0.5)\} = \emptyset$.
The worst optimal solution is $(x^{[6]}, y_1^{[6]}, y_2^{[6]}) = (0, 1, 2), f_1^r = 7$.

Therefore, the optimal value range is $[0.5, 7]$. The best optimal solution can be obtained by solving the problem $LB(c, d, a)$ with $c = 1, d = (1, 1)$ and $a = (2, -1)$. The worst solution can be obtained by solving the problem $LB(c, d, a)$ with $c = 4, d = (3, 2)$ and $a^* \in \{(a_1, a_2) : a_1 \in [2, 5], a_2 \in [-3, -1], 2a_1 + 3a_2 < 0\}$.

References

[1] P. Hansen, B. Jaumard, G. Savard, New branch-and-bound rules for linear bilevel programming, SIAM Journal on Scientific and Statistical Computing 13 (1992) 1194–1217.
[2] B. Colson, P. Marcotte, G. Savard, An overview of bilevel programming, Annals of Operations Research 153 (2007) 235–256.
[3] S. Dempe, Annotated bibliography on bilevel programming and mathematical programs with equilibrium constraints, Optimization 52 (2003) 333–359.
[4] L.N. Vicente, P.H. Calamai, Bilevel and multilevel programming: a bibliography review, Journal of Global Optimization 5 (1994) 291–306.
[5] J.F. Bard, Practical Bilevel Optimization: Algorithms and Applications, Kluwer Academic Publishers, Dordrecht, Boston, London, 1998.
[6] S. Dempe, Foundations of Bilevel Programming, Kluwer Academic Publishers, Dordrecht, Boston, London, 2002.
[7] K. Shimizu, Y. Ishizuka, J.F. Bard, Nondifferentiable and Two-Level Mathematical Programming, Kluwer Academic Publishers, Boston, London, Dordrecht, 1997.
[8] R.E. Steuer, Algorithms for linear programming problems with interval objective functions, Mathematics of Operations Research 6 (3) (1981) 333–348.
[9] H. Ishibuchi, H. Tanaka, Multiobjective programming in optimization of the interval objective function, European Journal of Operational Research 48 (2) (1990) 219–225.
[10] M. Inuiguchi, M. Sakawa, Minimax regret solution to linear programming problems with an interval objective function, European Journal of Operational Research 86 (3) (1995) 526–536.
[11] H.E. Mausser, M. Laguna, Minimising the maximum relative regret for linear programmes with interval objective function coefficients, Journal of the Operational Research Society 50 (10) (1999) 1063–1070.
[12] J.W. Chinneck, K. Ramadan, Linear programming with interval coefficients, Journal of the Operational Research Society 51 (2) (2000) 209–220.
[13] T. Shaocheng, Interval number and fuzzy number linear programming, Fuzzy Sets and Systems 66 (3) (1994) 301–306.
[14] M. Hladík, Optimal value range in interval linear programming, Fuzzy Optimization and Decision Making 8 (3) (2009) 283–294.
[15] M. Hladík, Generalized linear fractional programming under interval uncertainty, European Journal of Operational Research 205 (1) (2010) 42–46.
[16] M. Hladík, Optimal value bounds in nonlinear programming with interval data, TOP 19 (1) (2011) 93–106. doi:10.1007/s11750-009-0099-y.
[17] W.F. Bialas, M.H. Karwan, On two-level optimization, IEEE Transactions on Automatic Control 27 (1982) 211–214.
[18] G. Savard, Contribution à la programmation mathématique à deux niveaux, Ph.D. Thesis, Ecole Polytechnique de Montréal, Université de Montréal, Montréal, QC, Canada, 1989.
[19] M.S. Bazaraa, J.J. Jarvis, H.D. Sherali, Linear Programming and Network Flows, second ed., Wiley, New York, 1990.
[20] J.B.E. Etoa, Solving convex quadratic bilevel programming problems using an enumeration sequential quadratic programming algorithm, Journal of Global Optimization 47 (2010) 615–637.
[21] L.D. Muu, N.V. Quy, A global optimization method for solving convex quadratic bilevel programming problems, Journal of Global Optimization 26 (2) (2003) 199–219.
[22] L.N. Vicente, G. Savard, J. Judice, Descent approaches for quadratic bilevel programming, Journal of Optimization Theory and Applications 81 (1994) 379–399.
[23] S. Wang, Q. Wang, S. Romano-Rodríguez, Optimality conditions and an algorithm for linear-quadratic bilevel programs, Optimization 31 (1994) 127–139.
[24] H.I. Calvete, C. Galé, S. Dempe, S. Lohse, Bilevel problems over polyhedra with extreme point optimal solutions, Journal of Global Optimization (2011) in press (doi:10.1007/s10898-011-9762-6).